Exercise 1: [27 marks]

Consider the following linear programming problem

Maximize $Z = 4x_1 + 3x_2$

subject to

$$2x_1 + 4x_2 \le 2400 \quad +1 \qquad b_1$$

$$2x_1 + 3x_2 \le 1500$$

$$x_1 - x_2 \le 600$$

and $x_1 \ge 0,\ x_2 \ge 0$

1. **Find an optimal solution for the above given problem.**

Max $Z = 4x_1 + 3x_2$

$Z - 4x - 3x_2 = 0$ | 0.25

$2x_1 + 4x_2 + x_3 = 2400$ | 0.25

$2x_1 + 3x_2 + x_4 = 1500$ | 0.25

$x_1 - x_2 + x_5 = 600$ | 0.25

$x_i \ge 0,\ i = 1, 2, 3, 4, 5$

Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	Right Side
6 Z	1	-4	-3	0	0	0	0
x_3	0	2	4	1	0	0	2400 → 1200
x_4	0	2	3	0	1	0	1500 → 750
x_5	0	1	-1	0	0	1	600 → 600
initial BF(0, 0, 2400,	1500, 600)	0.25					
6 Z	1	0	-7	0	0	4	2400
x_3	0	0	6	1	0	-2	1200 → 200
x_4	0	0	5	0	1	-2	300 → 60
x_1	0	1	-1	0	0	1	600
BF(600, 0, 1200, 300, 0)	0.25						
4.75 Z	1	0	0	0	7/5	~~12/5~~	2820
x_3	0	0	~~1~~	1	~~-1~~	~~0~~	~~900~~
x_2	0	0	1	0	1/5	-2/5	60
x_1	0	1	0	0	1/5	3/5	660

BF(660, 60, 900, 0, 0)

The optimal solution is $(660, 60, \cancel{900}) \rightarrow Z = 2820$

0,8

Max $Z = 4x_1 + 3x_2$

$2x_1 + 4x_2 \leq 2400$

$2x_1 + 3x_2 \leq 1500 + D_2$

$x_1 - x_2 \leq 600$

new optimal :

$Z = 2820 + \frac{7}{5} D_2 + \frac{12}{5} D_3$

$x_3 = 900 + D_1 + D_2$

$x_2 = 60 + \frac{1}{5} D_2 - \frac{2}{5} D_3$

$x_1 = 600 + \frac{1}{5} D_2 + \frac{3}{5} D_3$

2. **Suppose that we want to increase the right-hand side of constraint 1 by 1. What can you conclude about the optimality of the solution obtained in question 1?**

$2x_1 + 4x_2 \leq 2401$

$2(660) + 4(60) \leq 2401$

$1320 + 240 \leq 2401$

$1560 \leq 2401$

∴ it still satisfy the constraint

∴ the solution is still optimal

Since the shadow price y_1^* of the first constraint is $= 0$

∴ b_1 is not sensetive

∴ changing in the Right hand-side of the first constraint will not effect on the objective function and the solution will remain optimal

1

3. **Find the allowable range for the right-hand side of constraint 2 over which the current optimal BF solution remains feasible. (Hint: consider the change in constraint 2 only)**

~~set $D_1 = 0$, $D_3 = 0$~~

Max $Z = 4x_1 + 3x_2$

$2x_1 + 4x_2 \leq 2400$
$2x_1 + 3x_2 \leq 1500 + D_2$
$x_1 - x_2 \leq 600$

2

new optimal

$Z = 2820 + \frac{7}{5} D_2$

$x_3 = 900 - D_2$ (0,5)
$x_2 = 60 + \frac{1}{5} D_2$
$x_1 = 600 + \frac{1}{5} D_2$ 660 (0,25)

the variables must be nonnegative

$x_3 = 900 - D_2 \geq 0$
$x_2 = 60 + \frac{1}{5} D_2 \geq 0$
$x_1 = 600 + \frac{1}{5} D_2 \geq 0$

(0,5)

$900 \geq D_2 \rightarrow D_2 \leq 900$
$\frac{1}{5} D_2 \geq -60 \rightarrow D_2 \geq -300$
$\frac{1}{5} D_2 \geq -600 \rightarrow D_2 \geq -3000$

$-300 \leq D_2 \leq 900$

$1200 \leq D_2 + 1500 \leq 2400$

$1200 \leq b_2 \leq 2400$ (0,25)

Exercise 2: [13 marks]

Consider the following linear programming problem

Minimize $Z = 3x_1 + 4x_2$

subject to

$$x_1 + 4x_2 \geq 8$$
$$2x_1 + 3x_2 \geq 12$$
$$2x_1 + x_2 \geq 6$$

and $x_1 \geq 0,\ x_2 \geq 0$

1. Find the dual of the above given problem.

Maximize $W = 8y_1 + 12y_2 + 6y_3$ 1

$y_1 + 2y_2 + 2y_3 \leq 3$ 1

$4y_1 + 3y_2 + y_3 \leq 4$ 1

$y_i \cancel{\leq} 0 \quad i = 1, 2, 3$

$\geq$

2. Use the fact that (0, 1.25, 0.25) is an optimal solution for the dual problem to find an optimal solution for the primal problem.

$y_1 \quad y_2 \quad y_3$

Augmented form of the dual problem:

$$\begin{cases} y_1 + 2y_2 + 2y_3 + y_4 = 3 \\ 4y_1 + 3y_2 + y_3 + y_5 = 4 \end{cases}$$ 0.5 0.5

$* \; 0 + 2(1.25) + 2(0.25) + y_4 = 3$

$y_4 = 3 - \frac{5}{2} - \frac{1}{2} = 0$ 0.25

$* \; y_5 = 4 - \frac{15}{4} - \frac{1}{4} = 0$ 0.25

$y_1 = 0, \; y_2 = 1.25, \; y_3 = 0.25, \; y_4 = 0, \; y_5 = 0$

$y_4 = z_1 - c_1$ $\qquad$ $(W = \frac{33}{2} = 16.5)$

$y_5 = z_2 - c_2$

Primal Var.		Dual Var.	
x_1	Basic	~~z_1-c_1~~ y_1	non-basic
x_2	Basic	~~z_2-c_1~~ y_2	non-basic
z_1-c_1 ~~x_3~~	Basic	y_1	non-basic
z_2-c_2 ~~x_4~~	non-basic	y_2	Basic
z_3-c_3 ~~x_5~~	non-basic	y_3	Basic

1

the Augmented form of the primal :

~~min Z = 3x_1 + 4x_2~~

$Z = 3x_1 + 4x_2$

$$\begin{cases} x_1 + 4x_2 + x_3 = 8 \\ 2x_1 + 3x_2 + x_4 = 12 \\ 2x_1 + x_2 + x_5 = 6 \end{cases}$$

$$\begin{array}{r} 2x_1 + 3x_2 = 12 \\ - \quad 2x_1 + x_2 = 6 \\ \hline 2x_2 = 6 \\ x_2 = 3 \end{array}$$

1

* $2x_1 + 3 = 6$

$2x_1 = 3 \rightarrow x_1 = \frac{3}{2}$ 1

* $\frac{3}{2} + 12 + x_3 = 8$

$x_3 = 8 - 12 - \frac{3}{2} = -\frac{11}{2}$

* $3 + 9 + x_4 = 12$

$x_4 = 0$

$Z = 3\left(\frac{3}{2}\right) + 4(3) = \frac{33}{2} = 16.5$

$\therefore W = Z = 16.5$

$\therefore$ using complemantry optimal solution property ~~we~~ the optimal solution for the primal is

$\left(\frac{3}{2}, 3\right) \qquad Z = 16.5$

1.25